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Norm continuity of solution semigroups of a class of neutral functional differential equations with distributed delay

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A B S T R A C T

In this note, we shall consider the norm continuity of a class of solution semigroups associated with linear functional differential equations of neutral type with time lag $r > 0$ in Hilbert spaces. The norm continuity plays an important role in the analysis of asymptotic stability of the system under consideration by means of spectrum approaches. We shall show that for a square integrable neutral delay term and an unbounded infinitesimal generator A multiplied by a square integrable weight function in the distributed delay term, the associated solution semigroup of the system is norm continuous at every $t > r$.

1. Introduction

For any Banach spaces X and Y, we always denote by $\mathscr{L}(X, Y)$ the space of all bounded, linear operators from X to Y. If $X = Y$, we simply write $\mathscr{L}(X, X)$ by $\mathscr{L}(X)$. Let V be a separable Hilbert space and V^* be its dual space. Suppose that $a : V \times V \to \mathbb{R}$ is a bounded bilinear form satisfying the so-called Gårding's inequality $a(x, x) \le -\alpha\|x\|_V^2$, $x \in V$, for some constant $\alpha > 0$. Let A be a linear operator defined by this form through $a(x, y) = \langle x, Ay\rangle_{V,V^*}, x,\ y \in V$, where $\langle\cdot,\cdot\rangle_{V,V^*}$ is the duality pair between V and V^*. Then $A \in \mathscr{L}(V, V^*)$ and A generates a C_0-semigroup e^{tA}, $t \ge 0$, on V^* such that $e^{tA} : V^* \to V$ for each $t > 0$. We introduce the Lions interpolation Hilbert space (see Tanabe [7]) $H = (V, V^*)_{1/2,2}$ between V and V^*, which is given by $H = \{x \in V^* : \int_0^\infty \|Ae^{tA}x\|_{V^*}^2 dt < \infty\}$ with its inner product

$$\langle x, y\rangle_H := \langle x, y\rangle_{V^*} + \int_0^\infty \langle Ae^{tA}x, Ae^{tA}y\rangle_{V^*} dt, x,\ y \in H. \tag{1.1}$$

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We identify the dual H^* of H with H, then it is easy to see that $V \hookrightarrow H = H^* \hookrightarrow V^*$ where the embedding $\hookrightarrow$ is dense and continuous with $\|x\|_H^2 \le \nu\|x\|_V^2$, $x \in V$, for some constant $\nu > 0$. Moreover, the semigroup e^{tA}, $t \ge 0$, generated by A is bounded and analytic on both H and V^* such that

$$\|e^{tA}\|_{\mathscr{L}(H)} \le 1, \|e^{tA}\|_{\mathscr{L}(V^*)} \le C, t \ge 0, \tag{1.2}$$

for some constant $C > 0$, and for any $T \ge 0$, there is a continuous embedding

$$L^2([0,T],V) \cap W^{1,2}([0,T],V^*) \subset C([0,T],H) \tag{1.3}$$

where $W^{1,2}([0,T],V^*)$ is the standard Sobolev space (see [3]).

Let $r > 0$ and $\mathcal{H} = H \times L^2([-r,0],V)$. Consider a retarded functional differential equation with distributed delay,

$$\begin{cases} dy(t) = Ay(t)dt + \displaystyle\int_{-r}^0 \beta(\theta)Ay(t+\theta)d\theta, t \ge 0, \\ y(0) = \phi_0,\ y(\theta) = \phi_1(\theta),\ \theta \in [-r,0],\ \phi = (\phi_0,\phi_1) \in \mathcal{H}, \end{cases} \tag{1.4}$$

where $\beta \in L^2([-r,0],\mathbb{C})$ and $\mathbb{C}$ is the space of all complex numbers. Note that since $\phi_1 \in L^2([-r,0],V)$, it generally does not make sense to talk about $\phi_0 = \phi_1(0)$ unless ϕ_1 satisfies more regular properties, e.g., ϕ_1 is a continuous function. As is well known, for any $\phi \in \mathcal{H}$ there exists a unique strong solution $y = y(t,\phi)$, $t \ge -r$, to (1.4). In particular, one can introduce in association with this solution a C_0-semigroup $\mathcal{S}(t)$, $t \ge 0$, on $\mathcal{H}$ by $\mathcal{S}(t)\phi = (y(t,\phi), y_t(\phi))$ for any $\phi \in \mathcal{H}$, where $y_t(\phi)(\theta) := y(t+\theta,\phi)$, $\theta \in [-r,0]$. As regards norm continuity of $\mathcal{S}(\cdot)$, i.e., $\mathcal{S}(\cdot) : [0,\infty) \to \mathscr{L}(\mathcal{H})$ is continuous in the uniform operator topology, Di Blasio et al. [1] have proved that if the weight function $\beta(\cdot)$ in the distributed delay term of (1.4) satisfies $\beta(\cdot) \in W^{1,2}([-r,0],\mathbb{C})$, then the associated solution semigroup $\mathcal{S}(t)$, $t \ge 0$, is a differentiable (thus, norm continuous) semigroup for $t > r$. Subsequently, Jeong in [2] has shown that if $\beta(\cdot)$ is Hölder continuous, i.e., for any θ, $\tau \in [-r,0]$, $|\beta(\theta) - \beta(\tau)| \le C|\theta - \tau|^\kappa$, $C > 0$, $\kappa \in (0,1]$, then the solution semigroup $\mathcal{S}(t)$ is norm continuous for $t > 3r$. In 2002, Mastinšek [5] further improved their results to obtain that when $\beta(\cdot) \in L^2([-r,0],\mathbb{C})$, the solution semigroup $\mathcal{S}(t)$, $t \ge 0$, is norm continuous for $t > r$.

In this work, we shall generalize the above case to consider the norm continuity of a class of neutral functional differential equations of the form,

$$\begin{cases} d\Big(y(t) - \displaystyle\int_{-r}^0 D(\theta)y(t+\theta)d\theta\Big) = A\Big(y(t) - \displaystyle\int_{-r}^0 D(\theta)y(t+\theta)d\theta\Big)dt + \displaystyle\int_{-r}^0 \beta(\theta)Ay(t+\theta)d\theta dt, t \ge 0, \\ y(0) = \phi_0 + \displaystyle\int_{-r}^0 D(\theta)\phi_1(\theta)d\theta,\ y(\theta) = \phi_1(\theta),\ \theta \in [-r,0],\ \phi = (\phi_0,\phi_1) \in \mathcal{H}, \end{cases} \tag{1.5}$$

where $D(\cdot) \in L^2([-r,0],\mathscr{L}(V))$ and $\beta(\cdot) \in L^2([-r,0],\mathbb{C})$. It was shown in Liu [4] that for each $\phi = (\phi_0,\phi_1) \in \mathcal{H}$, there exists a unique strong solution $y = y(t,\phi)$, $t \ge 0$, to Eq. (1.5) and for any $T \ge 0$, the solution y satisfies

$$\|y\|_{L^2([0,T],V)} + \|y\|_{W^{1,2}([0,T],V^*)} \le C(T)\|\phi\|_{\mathcal{H}}, \tag{1.6}$$

for some number $C(T) > 0$ which depends on $T \ge 0$. In association with this solution y to (1.5), it was further shown (see [4]) that one can introduce a C_0-semigroup $\mathcal{S}(t)$, $t \ge 0$, of (1.5) on $\mathcal{H}$ given by

$$\mathcal{S}(t)\phi = \Big(y(t,\phi) - \int_{-r}^0 D(\theta)y(t+\theta,\phi)d\theta, y_t(\phi)\Big), \quad t \ge 0, \phi \in \mathcal{H}. \tag{1.7}$$

In this short note, we shall establish the following result which will play an important role in the stability analysis of the system (1.5) (see, e.g., [1]).

Theorem 1.1. *Suppose that the condition*

$$D(\cdot) \in L^2([-r,0],\mathscr{L}(V)), \beta(\cdot) \in L^2([-r,0],\mathbb{C}), \tag{1.8}$$

holds, then the semigroup $\mathcal{S}(t)$, $t \geq 0$, defined in (1.7) is norm continuous for $t > r$.

2. Proof of Theorem 1.1

The proof will be given in several propositions. The idea is that, in order to prove the norm continuity of $\mathcal{S}(t)$, $t > r$, we will consider separately the operator norm continuity of each of the two components of $\mathcal{S}(t)$ in (1.7) under the condition (1.8).

Proposition 2.1. *Let $y \in L^2([-r,T],V)$ be a strong solution to* Eq. (1.5), *then for any $\varepsilon > 0$ and $0 < r < t < T$, the inequality*

$$\|y_{t+\delta}(\phi) - y_t(\phi)\|_{L^2([-r,0],V)} \leq \varepsilon\|\phi\|_{\mathcal{H}},$$

holds for every $\phi \in \mathcal{H}$ and $\delta > 0$ sufficiently small.

Proof. Let $t_0 > r$ and $\delta > 0$ sufficiently small. Since A is an isomorphism from V to V^* (see, Tanabe [6]), we have for the solution y to (1.5) and $t \in [t_0, T)$ that

$$\begin{aligned}\|y_{t+\delta}(\phi) - y_t(\phi)\|^2_{L^2([-r,0],V)} &= \int_{-r}^0 \|Ay(t+\delta+\theta) - Ay(t+\theta)\|^2_{V^*}d\theta = \int_{t-r}^t \|Ay(s+\delta) - Ay(s)\|^2_{V^*}ds \\ &\leq \int_{t_0-r}^T \|y(s+\delta) - y(s)\|^2_V ds = \|y(\cdot+\delta,\phi) - y(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)}.\end{aligned} \tag{2.1}$$

On the other hand, note that for $t \geq 0$, the strong solution y to (1.5) satisfies

$$\begin{aligned} y(t) &= e^{tA}\phi_0 + \int_{-r}^0 D(\theta)y(t+\theta)d\theta + \int_0^t e^{(t-s)A}\int_{-r}^0 \beta(\theta)Ay(s+\theta)d\theta ds \\ &= e^{tA}\phi_0 + \int_{-r}^0 D(\theta)y(t+\theta)d\theta + \int_0^t e^{sA}\int_{-r}^0 \beta(\theta)Ay(t-s+\theta)d\theta ds.\end{aligned} \tag{2.2}$$

Hence, for any sufficiently small $\delta > 0$, we obtain from (2.2) the following equality

$$\begin{aligned} y(t+\delta) - y(t) &= \Big(e^{(t+\delta)A}\phi_0 - e^{tA}\phi_0\Big) + \Big(\int_{-r}^0 D(\theta)(y(t+\delta+\theta) - y(t+\theta))d\theta\Big) \\ &\quad + \Big(\int_t^{t+\delta} e^{sA}\int_{-r}^0 \beta(\theta)Ay(t-s+\delta+\theta)d\theta ds\Big) \\ &\quad + \Big(\int_0^t e^{sA}\int_{-r}^0 \beta(\theta)A(y(t-s+\delta+\theta) - y(t-s+\theta))d\theta ds\Big) \\ &=: (I_1(t,\phi)) + (I_2(t,\phi)) + (I_3(t,\phi)) + (I_4(t,\phi)), t \geq 0.\end{aligned} \tag{2.3}$$

Now we estimate each item I_1, I_2, I_3 and I_4, respectively. Since A is an isomorphism from V to V^*, we have, by assumption and (1.1), that

$$\begin{aligned}\|I_1(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)} &= \int_{t_0-r}^T \|Ae^{(t+\delta)A}\phi_0 - Ae^{tA}\phi_0\|^2_{V^*}dt \\ &\leq \|e^{(t_0-r+\delta)A} - e^{(t_0-r)A}\|^2_{\mathscr{L}(V^*)}\int_{t_0-r}^\infty \|Ae^{(t-t_0+r)A}\phi_0\|^2_{V^*}dt \\ &\leq \|e^{(t_0-r+\delta)A} - e^{(t_0-r)A}\|^2_{\mathscr{L}(V^*)}\|\phi_0\|^2_H.\end{aligned} \tag{2.4}$$

As e^{tA}, $t \geq 0$, is analytic (thus, norm continuous) in $\mathscr{L}(V^*)$, it follows that for any $\varepsilon > 0$, when $\delta > 0$ is sufficiently small, $\|e^{(t_0-r+\delta)A} - e^{(t_0-r)A}\|^2_{\mathscr{L}(V^*)} < \varepsilon$, which, in addition to (2.4), further implies that $\|I_1(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)} \leq \varepsilon\|\phi_0\|^2_H \leq \varepsilon\|\phi\|^2_{\mathcal{H}}$, $\phi \in \mathcal{H}$.

For $I_2(\cdot)$, we extend the family of operators $D(\cdot)$ by defining $\tilde{D}(t) = D(t)$ for $t \in [-r,0]$ and $\tilde{D}(t) = 0$ for $t \notin [-r,0]$. Then, by using Hölder's inequality and (1.6), (2.3), we have

$$
\begin{aligned}
\|I_2(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)} &\leq \int_{t_0-r}^T \Big\|\int_{-r}^T (\tilde{D}(s-t-\delta) - \tilde{D}(s-t))y(s)ds\Big\|^2_V dt \\
&\leq \int_{t_0-r}^T \int_{-r}^T \|\tilde{D}(s-t-\delta) - \tilde{D}(s-t)\|^2_{\mathscr{L}(V)} dsdt \cdot \int_{-r}^T \|y(s,\phi)\|^2_V ds \\
&\leq TC(T) \int_{-\infty}^{\infty} \|\tilde{D}(t-\delta) - \tilde{D}(t)\|^2_{\mathscr{L}(V)} dt \cdot \|\phi\|^2_{\mathcal{H}},
\end{aligned}
\tag{2.5}
$$

for some $C(T) > 0$. Since $\tilde{D} \in L^2(\mathbb{R}, \mathscr{L}(V))$, we have that

$$
\lim_{\delta\to 0} \int_{-\infty}^{\infty} \|\tilde{D}(t-\delta) - \tilde{D}(t)\|^2_{\mathscr{L}(V)} dt = 0. \tag{2.6}
$$

Thus, for any $\varepsilon > 0$, by (2.5) and (2.6), it follows that

$$
\|I_2(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)} \leq \varepsilon\|\phi\|^2_{\mathcal{H}}, \tag{2.7}
$$

when $\delta > 0$ is sufficiently small.

For $I_3(\cdot,\phi)$, first note that since A is a closed operator, we have for $r < t_0 \leq t < T$ that $AI_3(t,\phi) = \int_{-\delta}^0 Ae^{(t-s)A} \int_{-r}^0 \beta(\theta)Ay(s+\delta+\theta)d\theta ds$. Recall that (see Tanabe [6]) $\|Ae^{(t-s)A}\|_{\mathscr{L}(V^*)} \leq M/(t-s) \leq M/t_0$, for all $s \in [-\delta, 0]$, for some constant $M > 0$. Hence, for $r < t_0 \leq t < T$, it follows by Hölder's inequality that

$$
\begin{aligned}
\|AI_3(t,\phi)\|_{V^*} &\leq \frac{M}{t_0} \int_{-\delta}^0 \int_{-r}^0 \|\beta(\theta)Ay(s+\delta+\theta)\|_{V^*} d\theta ds \\
&\leq \frac{M}{t_0}\sqrt{\delta}\Big\{\int_{-\delta}^0 \Big(\int_{-r}^0 \|\beta(\theta)Ay(s+\delta+\theta)\|_{V^*} d\theta\Big)^2 ds\Big\}^{1/2}.
\end{aligned}
\tag{2.8}
$$

On the other hand, we have by Hölder's inequality that

$$
\begin{aligned}
\int_{-\delta}^0 \Big(\int_{-r}^0 \|\beta(\theta)Ay(s+\delta+\theta)\|_{V^*} d\theta\Big)^2 ds &\leq \int_{-\delta}^0 \int_{-r}^0 |\beta(\theta)|^2 d\theta \cdot \int_{-r}^0 \|Ay(s+\delta+\theta)\|^2_{V^*} d\theta ds \\
&\leq \int_{-r}^0 |\beta(\theta)|^2 d\theta \cdot \delta \int_{-r}^{\delta} \|y(s)\|^2_V ds,
\end{aligned}
$$

which, together with (2.8) and the isomorphism of A from V to V^*, yields that

$$
\begin{aligned}
\|I_3(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)} &\leq \int_{t_0-r}^T \frac{M^2}{t_0^2}\delta\|\beta\|^2_{L^2([-r,0],\mathbb{C})} \int_{-r}^{\delta} \|y(s)\|^2_V dsdt \\
&\leq T\frac{M^2}{t_0^2}\delta\|\beta\|^2_{L^2([-r,0],\mathbb{C})}\|\phi_1\|^2_{L^2([-r,0],V)},
\end{aligned}
\tag{2.9}
$$

by virtue of the initial condition $y(t) = \phi_1(t)$ for $t \in [-r,0)$. By using the relations (1.3) and (2.9), we further obtain that for arbitrary $\varepsilon > 0$, $\|I_3(\cdot,\phi)\|_{L^2([t_0-r,T],V)} \leq \varepsilon\|\phi\|_{\mathcal{H}}$, when $\delta > 0$ is sufficiently small.

Now let us consider the last term $I_4(\cdot)$ in (2.3). We first define $\tilde{\beta}(t) = \beta(t)$ for $t \in [-r,0]$ and $\tilde{\beta}(t) = 0$ for $t \notin [-r,0]$. Then

$$
I_4(t,\phi) = \int_0^t e^{(t-s)A} \int_{-r}^T [\tilde{\beta}(u-s-\delta) - \tilde{\beta}(u-s)]Ay(u)duds, \quad t \in [t_0,T). \tag{2.10}
$$

Let $b(u) = \tilde{\beta}(u-\delta) - \tilde{\beta}(u)$, $u \in \mathbb{R}$. Then it follows that

$$\Big\| \int_{-r}^{T} [\tilde{\beta}(u-\cdot-\delta) - \tilde{\beta}(u-\cdot)]Ay(u)du \Big\|_{L^2([0,T],V^*)} \le \|b\|_{L^2(\mathbb{R})} \|y\|_{L^2([0,T],V)}, \tag{2.11}$$

where $\|b\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\tilde{\beta}(u-\delta) - \tilde{\beta}(u)|^2 du$. Since $\beta \in L^2([-r,0],\mathbb{C})$, we have $\lim_{\delta\to 0}\|b\|_{L^2(\mathbb{R})} = 0$. Thus, for any $\varepsilon > 0$, it follows from (1.6) and (2.11) that

$$\Big\| \int_{-r}^{T} [\tilde{\beta}(u-\cdot-\delta) - \tilde{\beta}(u-\cdot)]Ay(u)du \Big\|_{L^2([0,T],V^*)} \le \varepsilon \|\phi\|_{\mathcal{H}}, \tag{2.12}$$

when δ is sufficiently small. Further, by using the inequalities (2.10), (2.12) and Hölder's inequality we get

$$\begin{aligned} \|I_4(\cdot,\phi)\|^2_{L^2([t_0-r,T],V)} &\le \int_{t_0-r}^{T} \Big\| \int_0^t e^{(t-s)A} \int_{-r}^{T} [\tilde{\beta}(u-s-\delta) - \tilde{\beta}(u-s)]Ay(u)duds \Big\|_V^2 dt \\ &\le \int_0^T \int_0^t \|e^{(t-s)A}\|^2_{\mathscr{L}(V^*)} ds \int_0^t \Big\| \int_{-r}^{T} [\tilde{\beta}(u-s-\delta) - \tilde{\beta}(u-s)]Ay(u)du \Big\|^2_{V^*} dsdt \\ &\le \varepsilon C^2 T \|\phi\|^2_{\mathcal{H}}, \end{aligned} \tag{2.13}$$

where $C > 0$ is the constant in (1.2). Finally, by combining all the estimates $I_1(\cdot)$, $I_2(\cdot)$, $I_3(\cdot)$ and $I_4(\cdot)$ and using (2.3), we may conclude that for any $\varepsilon > 0$, the inequality

$$\|y(\cdot+\delta,\phi) - y(\cdot,\phi)\|_{L^2([t_0-r,T],V)} \le \varepsilon \|\phi\|_{\mathcal{H}}, \tag{2.14}$$

holds when $\delta > 0$ is small enough. Therefore, by (2.1) and (2.14) we have further that $\|y_{t+\delta}(\phi) - y_t(\phi)\|_{L^2([-r,0],V)} \le \varepsilon\|\phi\|_{\mathcal{H}}$ for every $r < t_0 \le t < T$ and $\delta > 0$ sufficiently small. Since t_0 $(> r)$ is arbitrary, the proof is thus complete. $\square$

Proposition 2.2. *Let $y \in L^2([-r,T],V)$ be a strong solution to* Eq. (1.5)*, then for any $\varepsilon > 0$ and $t \in (r,T)$,*

$$\Big\| \Big(y(t+\delta,\phi) - \int_{-r}^{0} D(\theta)y(t+\delta+\theta,\phi)d\theta \Big) - \Big(y(t,\phi) - \int_{-r}^{0} D(\theta)y(t+\theta,\phi)d\theta \Big) \Big\|_H \le \varepsilon\|\phi\|_{\mathcal{H}}, \tag{2.15}$$

hold for every $\phi \in \mathcal{H}$ and $\delta > 0$ sufficiently small.

Proof. We intend to give the estimate of the difference

$$\Big(y(t+\delta,\phi) - \int_{-r}^{0} D(\theta)y(t+\delta+\theta,\phi)d\theta \Big) - \Big(y(t,\phi) - \int_{-r}^{0} D(\theta)y(t+\theta,\phi)d\theta \Big)$$

in $L^2([t_0-r,T],V)$ and $W^{1,2}([t_0-r,T],V^*)$ norms, respectively. Since y satisfies Eq. (1.5), one can write the derivative of the above difference in the following form:

$$\begin{aligned} &\Big(y(t+\delta) - \int_{-r}^{0} D(\theta)y(t+\delta+\theta)d\theta \Big)'_t - \Big(y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta \Big)'_t \\ &\quad = A\Big(y(t+\delta) - \int_{-r}^{0} D(\theta)y(t+\delta+\theta)d\theta \Big) - A\Big(y(t) - \int_{-r}^{0} D(\theta)y(t+\theta)d\theta \Big) \\ &\qquad + \int_{-r}^{0} \beta(\theta)[Ay(t+\delta+\theta) - Ay(t+\theta)]d\theta, \end{aligned} \tag{2.16}$$

for almost every $t \in [0, T]$. Since A is an isomorphism from V to V^*, we get from the estimates (2.7) and (2.14) that

$$\begin{aligned}
&\Big\|A\Big(y(\cdot+\delta,\phi) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta)d\theta\Big) - A\Big(y(\cdot,\phi) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{L^2([t_0-r,T],V^*)} \\
&= \Big\|\Big(y(\cdot+\delta,\phi) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta,\phi)d\theta\Big) - \Big(y(\cdot,\phi) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{L^2([t_0-r,T],V)} \\
&\le \|y(\cdot+\delta,\phi) - y(\cdot,\phi)\|_{L^2([t_0-r,T],V)} + \|I_2(\cdot,\phi)\|_{L^2([t_0-r,T],V)} \le (\varepsilon + \sqrt{\varepsilon})\|\phi\|_{\mathcal{H}},
\end{aligned} \tag{2.17}$$

when $\delta > 0$ is small enough. On the other hand, we have by Hölder's inequality and (2.14) that

$$\begin{aligned}
&\int_{t_0-r}^T \Big(\int_{-r}^0 \|\beta(\theta)A(y(s+\delta+\theta) - y(s+\theta))\|_{V^*}d\theta\Big)^2 ds \\
&\quad\le \int_{t_0-r}^T \int_{-r}^0 |\beta(\theta)|^2 d\theta \cdot \int_{-r}^0 \|A(y(s+\delta+\theta) - y(s+\theta))\|^2_{V^*} d\theta ds \\
&\quad\le \int_{-r}^0 |\beta(\theta)|^2 d\theta \cdot \int_{-r}^T \|y(s+\delta) - y(s)\|^2_V ds \le \varepsilon\|\phi\|^2_{\mathcal{H}},
\end{aligned}$$

when $\delta > 0$ is small enough. Hence, from the above estimates and (2.16) we have that

$$\Big\|\Big(y(\cdot+\delta) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta)d\theta\Big) - \Big(y(\cdot) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{W^{1,2}([t_0-r,T],V^*)} \le \varepsilon\|\phi\|_{\mathcal{H}} \tag{2.18}$$

when $\delta > 0$ is small enough. On the other hand, by virtue of (2.7) and (2.14) we have that

$$\begin{aligned}
&\Big\|\Big(y(\cdot+\delta) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta)d\theta\Big) - \Big(y(\cdot) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{L^2([t_0-r,T],V)} \\
&\le \|y(\cdot+\delta) - y(\cdot)\|_{L^2([t_0-r,T],V)} + \|I_2(\cdot,\phi)\|_{L^2([t_0-r,T],V)} \le \varepsilon\|\phi\|_{\mathcal{H}} + \sqrt{\varepsilon}\|\phi\|_{\mathcal{H}},
\end{aligned} \tag{2.19}$$

which, together with (2.17), immediately yields that for any $\varepsilon > 0$,

$$\begin{aligned}
&\Big\|\Big(y(\cdot+\delta) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta)d\theta\Big) \\
&\quad - \Big(y(\cdot) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{L^2([t_0-r,T],V)\cap W^{1,2}([t_0-r,T],V^*)} \le \varepsilon\|\phi\|_{\mathcal{H}},
\end{aligned} \tag{2.20}$$

when $\delta > 0$ is sufficiently small. Hence, by the embedding (1.3) and (2.20), there exists a constant $C > 0$ such that

$$\begin{aligned}
&\Big\|\Big(y(\cdot+\delta) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta)d\theta\Big) - \Big(y(\cdot) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{C([t_0-r,T],H)} \\
&\le C\Big\|\Big(y(\cdot+\delta) - \int_{-r}^0 D(\theta)y(\cdot+\delta+\theta)d\theta\Big) \\
&\quad - \Big(y(\cdot) - \int_{-r}^0 D(\theta)y(\cdot+\theta)d\theta\Big)\Big\|_{L^2([t_0-r,T],V)\cap W^{1,2}([t_0-r,T],V^*)} \le C\varepsilon\|\phi\|_{\mathcal{H}}.
\end{aligned} \tag{2.21}$$

In particular, for any $t \in [t_0, T)$ and $\varepsilon > 0$, when $\delta > 0$ is sufficiently small, we have

$$\Big\|\Big(y(t+\delta) - \int_{-r}^0 D(\theta)y(t+\delta+\theta)d\theta\Big) - \Big(y(t) - \int_{-r}^0 D(\theta)y(t+\theta)d\theta\Big)\Big\|_H \le \varepsilon\|\phi\|_{\mathcal{H}}. \tag{2.22}$$

The proof is thus complete. □

Proof of Theorem 1.1. The whole proof is a consequence of Propositions 2.1 and 2.2. More precisely, let $r < t_0 \leq t < T$, we have from the relevant estimates in Propositions 2.1 and 2.2 that for any $\varepsilon > 0$,

$$\begin{aligned}\|\mathcal{S}(t+\delta)\phi - \mathcal{S}(t)\phi\|^2_{\mathcal{H}} = \bigg\| &\Big(y(t+\delta,\phi) - \int_{-r}^0 D(\theta)y(t+\delta+\theta,\phi)d\theta\Big) \\ &- \Big(y(t,\phi) - \int_{-r}^0 D(\theta)y(t+\theta,\phi)d\theta\Big)\bigg\|^2_H + \|y_{t+\delta}(\phi) - y_t(\phi)\|^2_{L^2([-r,0],V)} \leq \varepsilon\|\phi\|^2_{\mathcal{H}}\end{aligned}$$

holds for every $\phi \in \mathcal{H}$ and sufficiently small $\delta > 0$. This shows the uniform continuity of t on a closed interval $[t_0, T]$, $t_0 > r$. Therefore, the solution semigroup $\mathcal{S}(t)$, $t \geq 0$, is norm continuous at every $t > r$ since t_0 $(>r)$ is arbitrary. The proof is thus complete.

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